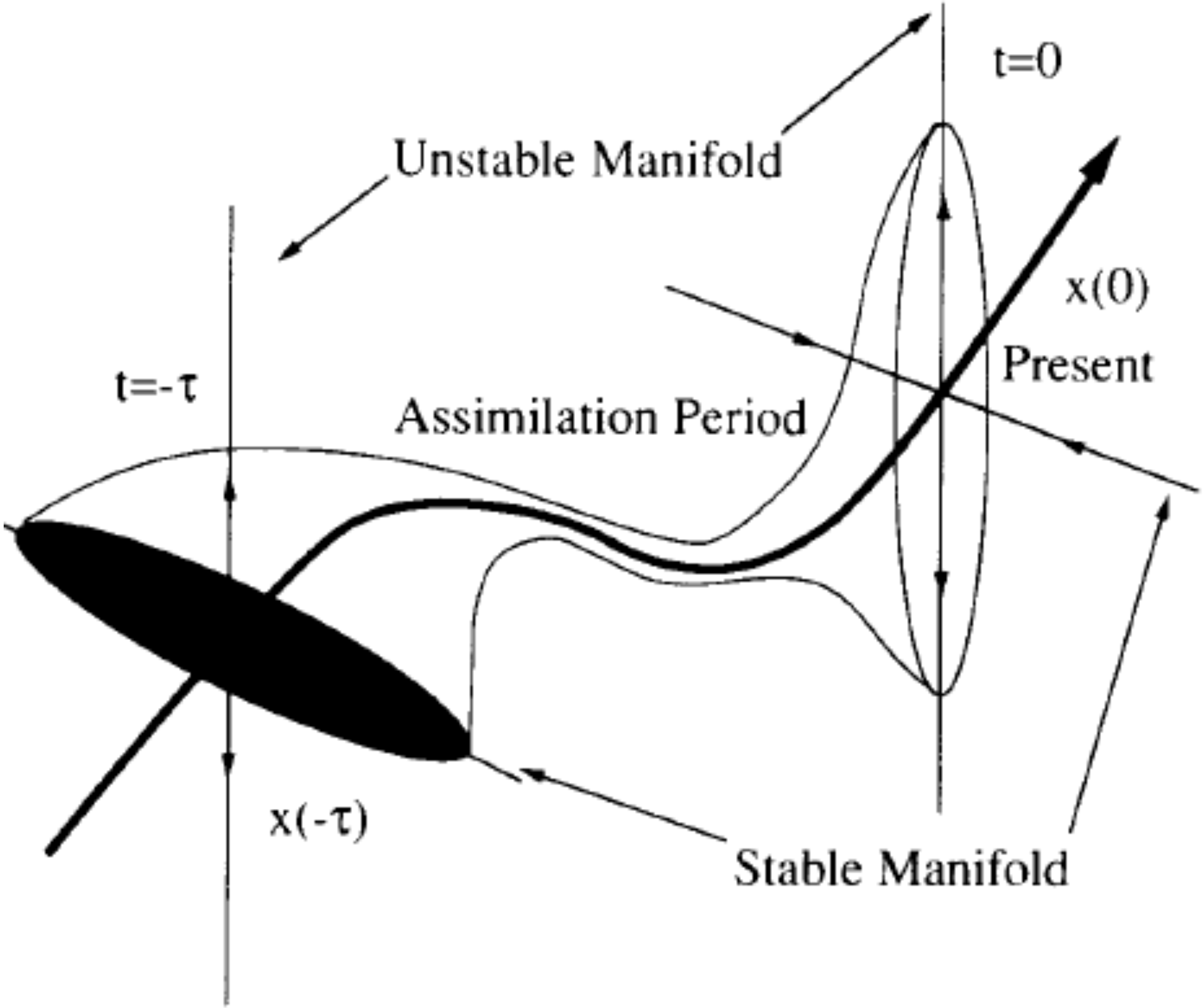
If there is uncertainty on the state of the system, and dynamics of the system is perfectly known, uncertainty on the state along stable modes decreases over time, while uncertainty along unstable modes increases.

Stable (unstable) modes : perturbations to the basic state that decrease (increase) over time.



t=0
Unstable Manifold
x(0)
Present
t=-τ
Assimilation Period
x(-τ)
Stable Manifold

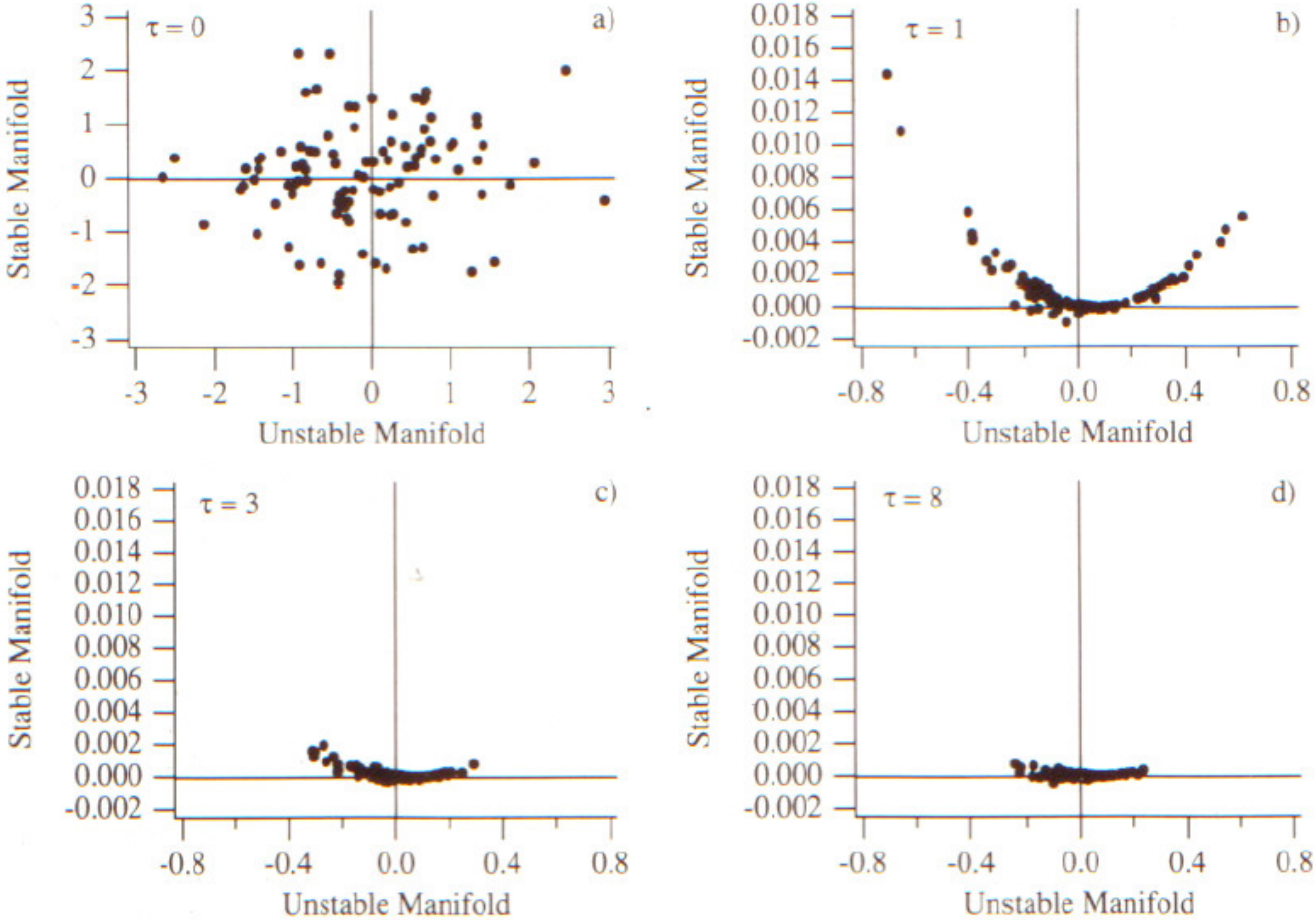


Fig. 7. Projection of the 100 minimizing solutions, at the end of the assimilation period, onto the plane spanned by the stable and unstable directions, defined as in Fig. 3. Values of τ are indicated on the panels. The projection is not an orthogonal projection, but a projection parallel to the local velocity vector $(\mathrm{d}x/\mathrm{d}t, \mathrm{d}y/\mathrm{d}t, \mathrm{d}z/\mathrm{d}t)$ (central manifold).

Pires *et al., Tellus*, 1996 ; Lorenz system (1963)

Since, after an assimilation has been performed over a period of time, uncertainty is likely to be concentrated in modes that have been unstable, it might be useful, at least in terms of cost efficiency, to concentrate assimilation in modes that have been unstable in the recent past, where uncertainty is likely to be largest.

Also, presence of residual noise in stable modes can be damageable for analysis and subsequent forecast.

Assimilation in the Unstable Subspace (*AUS*) (Carrassi *et al*., 2007, 2008, for the case of 3D-Var)

Four-dimensional variational assimilation in the unstable subspace (*4DVar-AUS*)

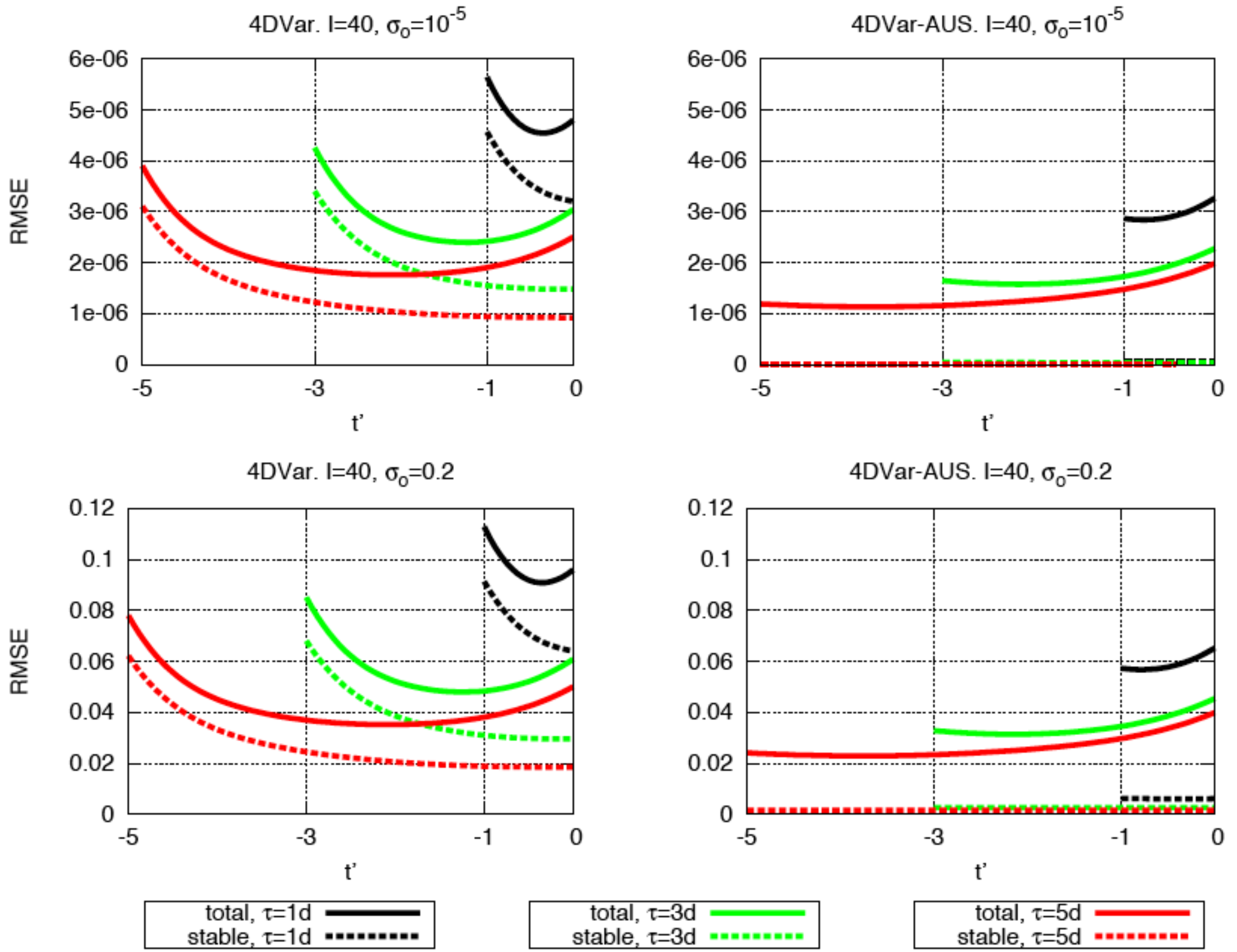


Figure 3. Time average RMS error within 1, 3, 5 days assimilation windows as a function of $t' = t - \tau$, with $\sigma_o = .2, 10^{-5}$ for the model configuration $I = 40$. Left panel: 4DVar. Right panel: 4DVar-AUS with $N = 15$. Solid lines refer to total assimilation error, dashed lines refer to the error component in the stable subspace $\mathbf{e}_{16}, \ldots, \mathbf{e}_{40}$.

Trevisan *et al*., 2010, Lorenz 96 system

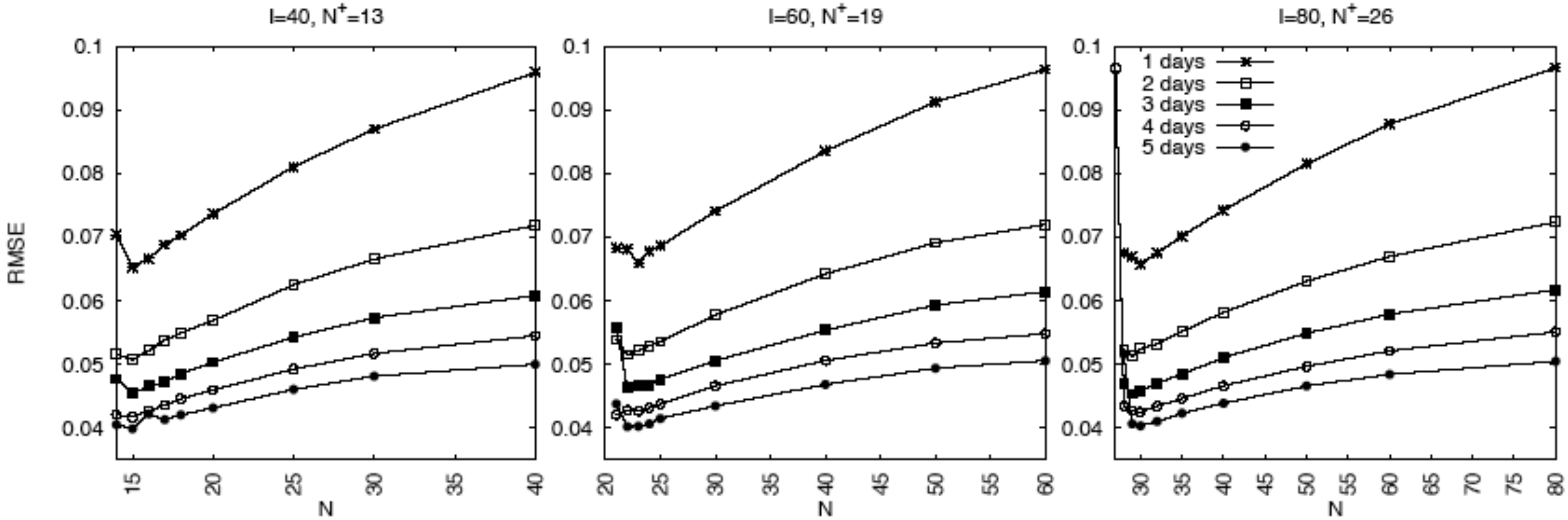


Figure 1. Time average RMS analysis error at $t = \tau$ as a function of the subspace dimension N for three model configurations: I=40, 60, 80. Different curves in the same panel refer to different assimilation windows from 1 to 5 days. The observation error standard deviation is $\sigma_o = 0.2$.

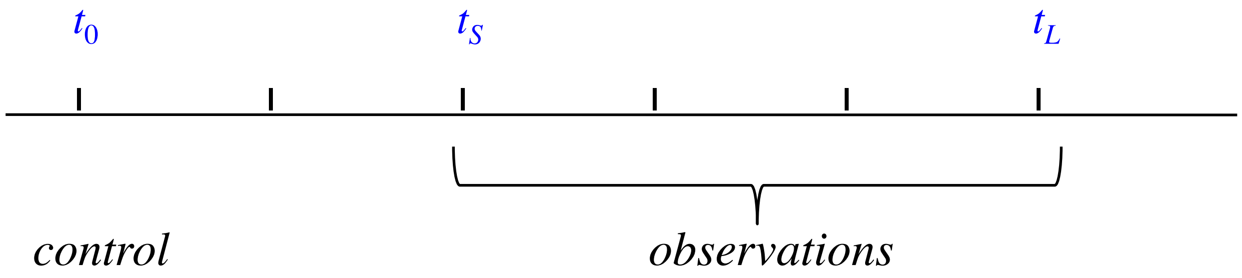
No explicit background term (*i. e.*, with error covariance matrix) in objective function : information from past lies in the background to be updated, and in the N perturbations which define the subspace in which updating is to be made.

Values I = 40, 60, 80, correspond respectively to N^+ = 13, 19 and 26 positive Lyapunov exponents.

Best performance for N slightly above number N^+ of positive Lyapunov exponents.

Iterative Ensemble Kalman Smoother (*IEnKS*, Bocquet and Sakov, 2014)

Minimization performed at time t_0, in an appropriately chosen reduced subspace, assimilating observations performed between times t_S and t_L, with $t_0 \leq t_S \leq t_L$



If the dimension of the reduced subspace is small enough, gradient of objective function can be computed by finite differences, and approximate Hessian can be determined. Once the minimization has been achieved, a new ensemble of perturbations can be obtained by transport of the approximate inverse Hessian.

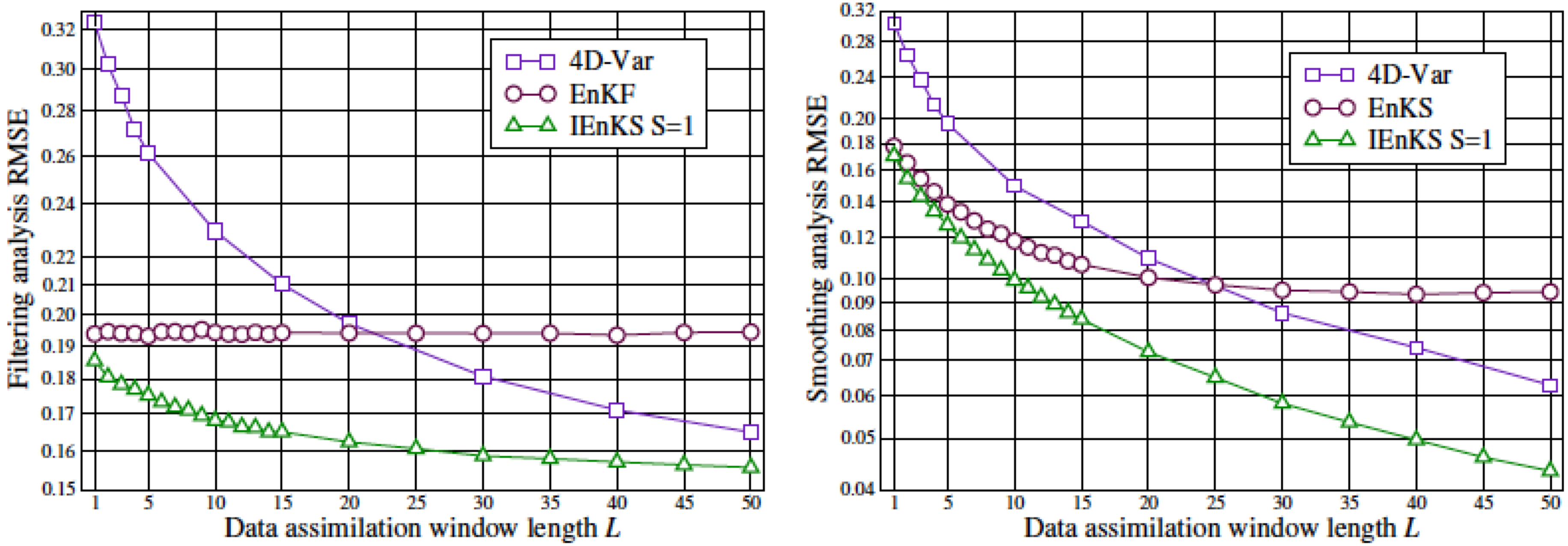


Figure 5: Average root mean square error of several DA methods computed from synthetic experiments with the Lorenz-96 model. The left panel shows the filtering analysis root mean square error of optimally tuned EnKF, 4DVar, IEnKS assimilation experiments, as a function of the length of the DAW. The right panel shows the smoothing analysis root mean square error of optimally tuned EnKS, 4DVar and IEnKS as a function of the length of their data assimilation window. The optimal RMSE is chosen within the window for 4DVar and it is taken at the beginning of the window for the IEnKS. The EnKF, EnKS and IEnKS use an ensemble of $N = 20$, which avoids the need for localization but requires inflation. The length of the DAW is $L \times \Delta t$, where $\Delta t = 0.05$.

Carrassi *et al.*, 2018

All schemes that have been presented so far are ‘Gaussian’, in the sense that they are more or less empirical and heuristic extensions, to moderately non-linear and non-Gaussian situations, of algorithms which achieve Bayesian estimation in linear and Gaussian situations.

Used not for theoretical reasons, but for purely pragmatic ones. They are the simplest of non simplicist algorithms, and they work.

Exact bayesian estimation ?

Particle filters

Predicted ensemble at time t : $\{x^b{}_l, l = 1, \ldots, L\}$, each element with its own weight (probability) $P(x^b{}_l)$

Observation vector at same time : $y = H(x) + \varepsilon$

Bayes' formula

$$P(x^b{}_l|y) = P(y|x^b{}_l)\ P(x^b{}_l)\ /\ P(y)$$

Defines updating of weights

Bayes' formula

$$P(x^b{}_l|y) \sim P(y|x^b{}_l)\ P(x^b{}_l)$$

If error ε is independent of all previous data

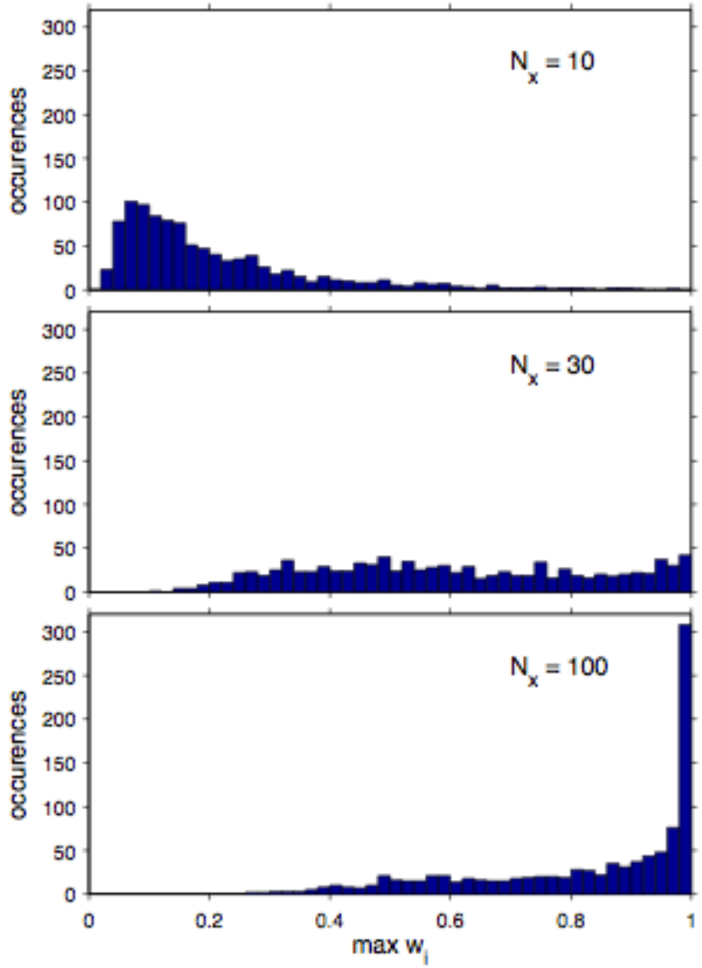
$$y = H(x) + \varepsilon \qquad \Rightarrow \qquad P(y|x^b{}_l) = P[\varepsilon = y - H(x^b{}_l)]$$

Defines updating of weights; particles are not modified. Asymptotically converges to bayesian pdf. Very easy to implement.

Observed fact. For large state dimension, ensemble tends to collapse.

Behavior of $\max w^i$

- $N_e = 10^3$; $N_x = 10, 30, 100$; 10^3 realizations



average squared error of posterior mean = 5.5

... = 25

... = 127

C. Snyder, http://www.cawcr.gov.au/staff/pxs/wmoda5/Oral/Snyder.pdf

Problem originates in the ‘curse of dimensionality’. Large dimension pdf’s are very diffuse, so that very few particles (if any) are present in areas where conditional probability (‘*likelihood*’) $P(y|x)$ is large.

Bengtsson *et al*. (2008) and Snyder *et al*. (2008) evaluate that stability of filter requires the size of ensembles to increase exponentially with space dimension.

Solution. *Resampling*. Remove particles with low likelihood, and create new particles in regions in state space with high likelihood, including using future observations for evolving particles between successive analysis times.

Review in van Leeuwen, 2017, *Annales de la faculté des sciences de Toulouse Mathématiques*

The Equivalent-Weights Particle Filter (Ades and van Leeuwen, *QJRMS*, 2013).

Example

Vorticity equation model with stochastic perturbations.

State-vector dimension ≈ 65,000

Decorrelation time: 25 timesteps

One complete noisy model field observed every 50 timesteps

24 particles.

Results after 12 analyses

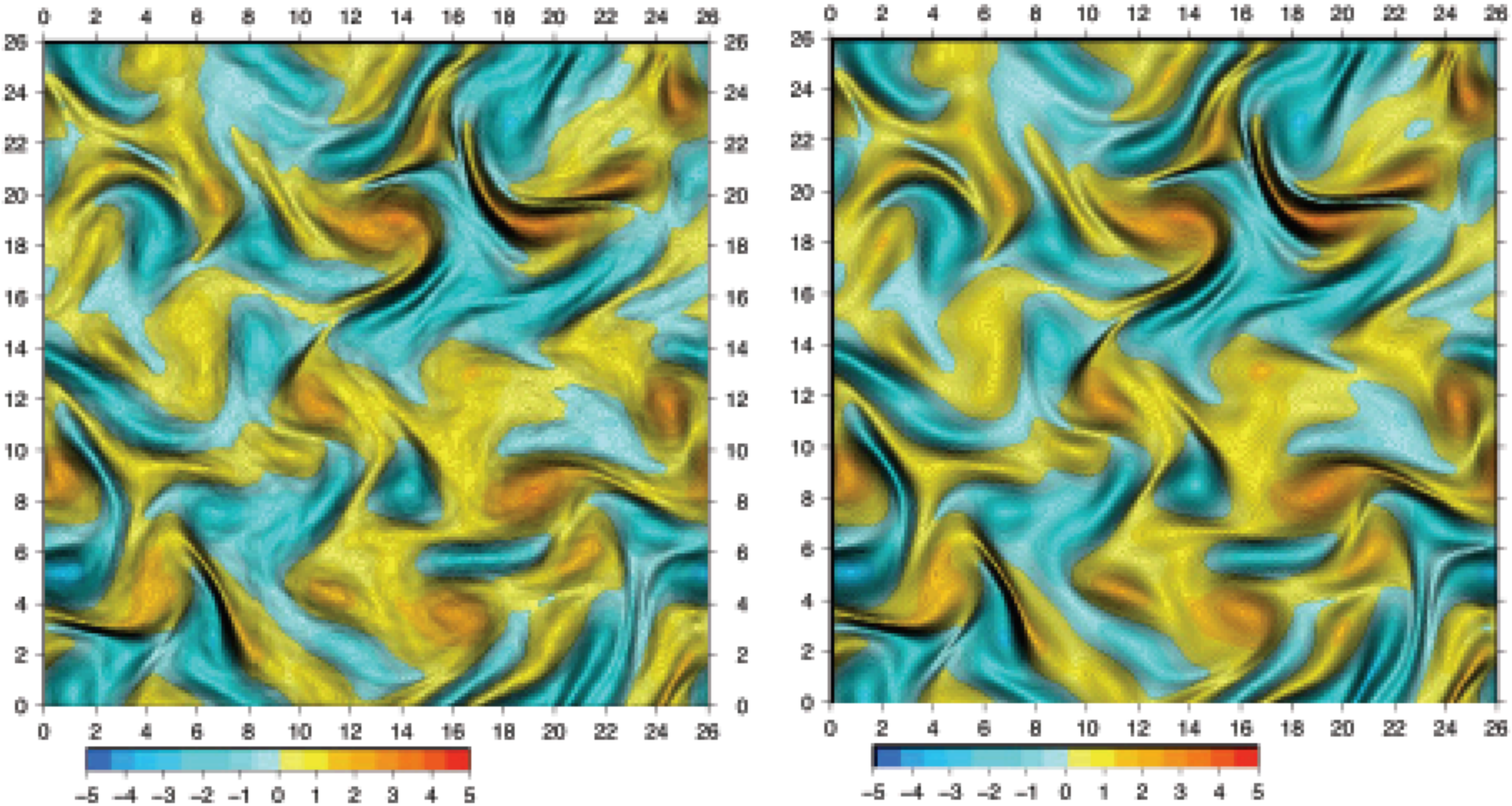


Figure 5.3. Snap shot of the vorticity field of the truth (right) and the particle filter mean (left) at time 25. Note the highly chaotic state of the fields, and the close to perfect tracking.

van Leeuwen, 2017

Bayesianity : filters are bayesian (in the limit of infinite ensemble size)

Possible difficulties : numerical implementation, numerical cost

Particle filters are actively studied (van Leeuwen, Morzfeld, …)

Assimilation, which originated from the need of defining initial conditions for numerical weather forecasts, has gradually extended to many diverse applications, first in climate and environmental science, and then in other domains

- Oceanography
- Atmospheric chemistry (both troposphere and stratosphere)
- Terrestrial biosphere and vegetation cover
- Reassimilation of past observations (mostly for climatological purposes, ECMWF, NCEP/NCAR)
- Definition of observing systems *(Observing Systems Simulation Experiments)*
- Planetary atmospheres (Mars, …)

- Magnetism (both planetary and stellar)
- Plate tectonics

- Parameter identification
- Validation of models
- Economics

- Dynamics of Covid-19 pandemic (Evensen *et al*., 2021)

- Mathematical studies, independently of direct real life applications
- …

It has now become a major tool of numerical environmental science, and beyond

A few of the (many) remaining problems :

- Observability (what to observe in order to know what we want to know ? Data are noisy, system is chaotic !)

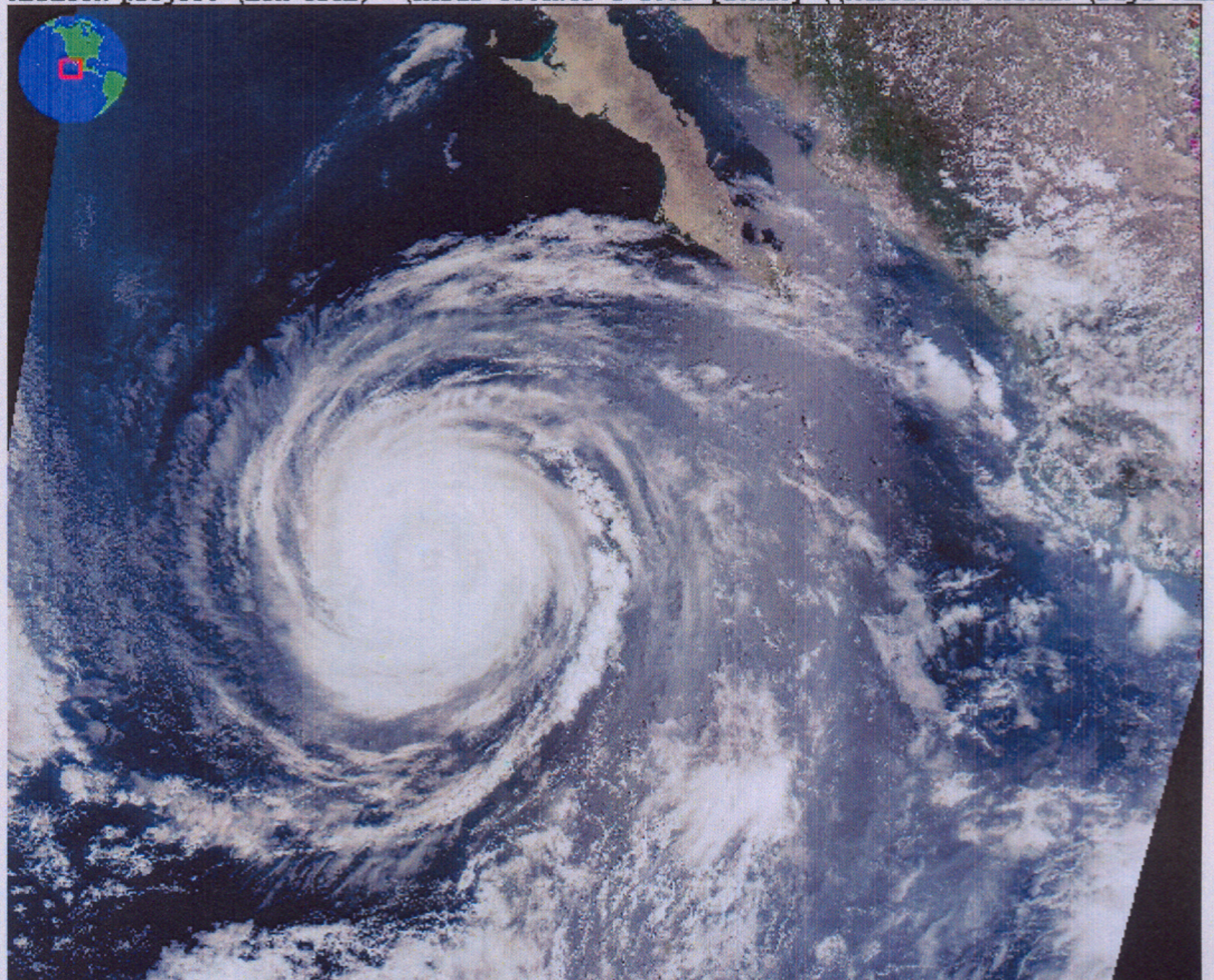
- More accurate identification and quantification of errors affecting data particularly the assimilating model (will always require independent hypotheses)

- Assimilation of images

- Algorithmics (Artificial Intelligence, Machine Learning, for instance for identification of sub-grid scale processes or observation operators)

- …

. HDFLook project (LOA-USTL) (MODIS October 2 2002 [18h10] ((Hurricane Hernan (Baja Cali



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Thanks !